

A Theorem of Probability

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Abstract

We prove here the above titled theorem the applications of which will be given elsewhere.

Key words ; almost sure convergence.

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Let (Ω, \mathcal{F}, P) be a probability space and $X_n := X_n(\omega)$, $(n = 1, 2, 3, \dots)$ be its random variables with $X_n(\omega) \geq 0$ for $\forall \omega \in \Omega$.

$$E[X] := E[X(\omega)] := \int_{\Omega} X(\omega) dP(\omega)$$

denotes the expectation value(or mean) of the random variable $X = X(\omega)$.

We put $\{K_m\}_{m=1}^{\infty}$ to be a natural number sequence with $K_m < K_{m+1}$ and $K_m \rightarrow +\infty$ (as $m \rightarrow +\infty$).

Then we have

Theorem 1 *If*

$$E\left[\sum_{m=1}^{\infty} X_{K_m+l_m}(\omega)\right] < +\infty \text{ for } 0 < \forall l_m \leq K_{m+1} - K_m, \quad (1)$$

then we have

$$X_n(\omega) \rightarrow 0 \text{ almost surely on } \Omega \text{ (as } n \rightarrow \infty). \quad (2)$$

proof The conclusion of the theorem is equivalent to

$$P\{\omega \in \Omega \mid \forall \epsilon > 0, \exists N \equiv N(\epsilon, \omega), X_n(\omega) \leq \epsilon \text{ for } \forall n \geq N\} = 1 \quad (3)$$

Firstly we assume our denying of this conclusion, that is, we assume

$$P\{\omega \in \Omega \mid \exists \epsilon(\omega) > 0, \forall N \in \mathbf{N}, \exists n \equiv n(\omega) \geq N, X_n(\omega) > \epsilon(\omega)\} > 0 \quad (4)$$

which will lead to a contradiction later.

(4) is equivalent to $P\{A\} > 0$ with

$$A := \{\omega \in \Omega \mid \exists \epsilon(\omega) > 0, \exists \{n_j \equiv n_j(\omega)\}_{j=1}^\infty, n_j \rightarrow \infty, X_{n_j}(\omega) > \epsilon(\omega)\} \quad (5)$$

We put the followings:

$$B_n := \{\omega \in \Omega \mid X_n(\omega) > \epsilon(\omega)\} \quad (6)$$

$$\{m_j\}_{j=1}^\infty := \{n \in \mathbf{N} \mid P\{B_n\} > 0\} \text{ with } m_j < m_{j+1} \quad (7)$$

$$C_N := \bigcup_{m_j \geq N} B_{m_j} \quad (8)$$

Then we have trivially

$$C_N \supset C_{N+1} \quad (9)$$

which shows the existence of

$$C := \lim_{N \rightarrow \infty} C_N = \limsup_{j \rightarrow \infty} B_{m_j} \quad (10)$$

and

$$C = A \text{ except } P\text{-measure zero set} \quad (11)$$

Next we divide $\{m_j\}_{j=1}^\infty$ into $\{m_j(k)\}_{j=1}^\infty$'s as follows:

$$\{m_j\}_{j=1}^\infty = \bigcup_{k \in \Lambda} \{m_j(k)\}_{j=1}^\infty \quad (12)$$

with

$$m_j(k) < m_{j+1}(k) \quad (13)$$

$$\{m_j(k)\}_{j=1}^\infty \cap \{m_j(l)\}_{j=1}^\infty = \emptyset (k \neq l) \quad (14)$$

$$\#\{\{m_j(k)\}_{j=1}^\infty \cap (K_m, K_{m+1}]\} \leq 1 \text{ for } \forall m \in \mathbf{N} \quad (15)$$

$$\#\Lambda = \#\mathbf{N} (\text{i.e. countably many}) \quad (16)$$

where $\#A$ denotes the number of elements of the set A .
We also put

$$D_N^{(k)} := \bigcup_{m_j(k) \geq N} B_{m_j(k)}. \quad (17)$$

Then we also have trivially

$$D_N^{(k)} \supset D_{N+1}^{(k)} \quad (18)$$

which leads to the existence of

$$D^{(k)} = \lim_{N \rightarrow \infty} D_N^{(k)} = \limsup_{j \rightarrow \infty} B_{m_j(k)} \quad (19)$$

and

$$C_N = \bigcup_{k \in \Lambda} D_N^{(k)}. \quad (20)$$

From (20), it follows that when N tends to ∞

$$A = C = \bigcup_{k \in \Lambda} D^{(k)} \quad \text{except } P - \text{measure zero set} \quad (21)$$

which leads to

$$\exists l \in \mathbf{N} \quad \text{such that} \quad P\{D^{(l)}\} > 0 \quad (22)$$

because of $P\{A\} > 0$ and $\#\Lambda = \#\mathbf{N}$.

We put

$$K := \int_{D^{(l)}} \epsilon(\omega) dP(\omega). \quad (23)$$

Because of the assumption of the theorem (1), that is,

$$\mathbb{E}[\sum_{j=1}^{\infty} X_{m_j(l)}(\omega)] < +\infty \quad (24)$$

there exists a natural number N such that

$$\frac{K}{2} > \mathbb{E}[\sum_{m_j(l) \geq N} X_{m_j(l)}(\omega)] \quad (25)$$

$$= \sum_{m_j(l) \geq N} \int_{\Omega} X_{m_j(l)}(\omega) dP(\omega) \quad (26)$$

$$> \sum_{m_j(l) \geq N} \int_{B_{m_j(l)}} X_{m_j(l)}(\omega) dP(\omega) \quad (27)$$

$$> \sum_{m_j(l) \geq N} \int_{B_{m_j(l)}} \epsilon(\omega) dP(\omega) \quad (28)$$

$$\geq \int_{D_N^{(l)}} \epsilon(\omega) dP(\omega) \quad (29)$$

due to $D_N^{(l)} = \bigcup_{m_j(l) \geq N} B_{m_j(l)}$. Because of

$$\int_{D_N^{(l)}} \epsilon(\omega) dP(\omega) \geq \int_{D_{N+1}^{(l)}} \epsilon(\omega) dP(\omega) \rightarrow \int_{D^{(l)}} \epsilon(\omega) dP(\omega) = K, \quad (30)$$

we have

$$\int_{D_N^{(l)}} \epsilon(\omega) dP(\omega) \geq K \quad (31)$$

with sufficiently large N . From (25), \dots , (31), we have a contradiction:

$$\frac{K}{2} \geq K. \quad (32)$$

Therefore we cannot have (4) or (5) which means the conclusion of the theorem: (2) or (3). This completes the proof.

References

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